NPS55GV72111A

NAVAL POSTGRADUATE SCHOOL

Monterey, California

METHODS FOR ASSESSING VARIABILITY, WITH EMPHASIS ON SIMULATION DATA INTERPRETATION

by

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November 1972

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ABSTRACT

This report describes and illustrates the use of a grouping technique (the jackknife) for setting confidence limits in simulation situations.

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Executive Summary

METHODS FOR ASSESSING VARIABILITY, WITH EMPHASIS ON SIMULATION DATA INTERPRETATION

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Many problems in operations research involve the use of a rather small sample of data to estimate certain features of an unknown probability distribution. For example, one might wish to estimate the expected waiting time for a repair process, and the variance or standard deviation of that waiting time, from a sample of, say, $n = 10$ observations. Then one might wish to assess the uncertainty in these estimates. If simulation (Monte Carlo sampling) techniques are used, small samples often arise because of the desire to utilize computer time efficiently.

This report describes the results of experiments with a relatively new procedure (called the "jackknife" by John Tukey) for estimation, and assessment of the variability of statistical estimates. The procedure is an improvement over standard procedures, but it is anticipated that further work will sharpen our jackknife, and furnish it with more blades.

* This paper was written under contracts with the Office of Naval Research and the Naval Personnel Research and Development Laboratories, Washington, D.C.

METHODS FOR ASSESSING VARIABILITY, WITH EMPHASIS ON SIMULATION DATA INTERPRETATION

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1. Introduction

Many of the stochastic quantities that arise in operations research, and in hydrology in particular, have distributions that are distinctly non-normal (non-Gaussian) in appearance. For example, Shane and Lynn in [7] have fitted hydrological variables by exponential distributions, and this model has subsequently been used in decision analysis studies. Of course there is little hope of showing, either theoretically or from existing data, that such random variables, or others derived from combinations thereof, are truly exponential, or in fact are members of any particular, simple, parametric family. Thus it is of interest to develop statistical methods that are applicable to rather general families of distributions, giving inferential methods that stand a better chance of being broadly valid than is true of the usual normal-distribution-calibrated procedures, and yet are apt to be more powerful than strictly non-parametric methods.

In this paper we investigate the performance of one useful method, the jackknife of Quenouille and Tukey, for setting confidence limits on the variance or standard deviation of an unknown distribution. In other words, we describe a method for estimating the spread or scale of the distribution of interest, and then estimating the error of our estimate. We also show by empirical sampling using a computer that the jackknife method seems to be an improvement upon normal-theory methods.

Empirical sampling appears to be the only method available for such studies when sample sizes are small, as will typically be true in operations research or in systems studies.

Finally, we discuss the performance of the jackknife method for analyzing a stochastic simulation experiment. The simulation makes use of the Monte Carlo technique known as antithetic variates; this technique is introduced in order to improve simulation efficiency.

2. What is the Jackknife?

Suppose that one obtains a set of observations $x_1, x_2, x_3, \ldots, x_n$ that may be assumed to come independently from a fixed, but unknown, probability distribution. In practice, n may be small: somewhere between ten and fifty. Even if a longer series is available, it may be profitable to split it into shorter pieces, for example, in order to check for temporal or spatial homogeneity.

Under the assumptions, it is well-known that if the location parameter or mean of the distribution is estimated by $\bar{x}$ (not necessarily a good idea; see for example recent studies of robustness by Andrews, et al, [1]), then the variance of the unknown distribution is estimated by

$$s^2 = \frac{1}{n-1} \sum_{i=1}^{n} (x_i - \bar{x})^2 \qquad (2.1)$$

and the variance of $\bar{x}$ by $\frac{s^2}{n}$ Student's t, or a "robustified" version thereof, then furnishes at least approximate confidence limits for the unknown mean. Evidence attests to the surprising validity of two-sided confidence limits so constructed.

Because observations are assumed to be independent, and because $\bar{x}$ is a linear combination of those observations, we are able to estimate the variability of $\bar{x}$ in repeated samples by making use of the same observations that enter $\bar{x}$. However, if we now wish to estimate the error with which we estimate the population variance via s^2, greater difficulties arise. The usual text-book approach to confidence limits utilizes the fact that in normal (Gaussian) samples $\frac{(n-1)s^2}{\sigma^2}$ has the chi-squared distribution on $n-1$ degrees of freedom. Unfortunately, confidence limits constructed using this fact are apt to be invalid when the underlying distribution is non-normal. If in fact the observations come from an exponential-like skewed distribution, the chi-squared confidence limits tend to be much too short, and we are led to attribute unjustified precision to our estimate.

One approach to this problem is some sort of grouping technique. The most obvious is to split the sample into k disjoint subsamples, each of size r $(kr=n)$, and then compute $s_1^2, s_2^2, \ldots, s_k^2$, i.e. k independent estimates of the variance based on $r-1$ degrees of freedom. One can then average these latter to obtain an estimate of the population variance, σ^2:

$$\sigma_{est}^2 = \frac{1}{k} \sum_{i=1}^{k} s_i^2 = \overline{s^2} \tag{2.2}$$

and the variance of σ_{est}^2 may be estimated by

$$\text{Var}\,[\sigma_{est}^2]_{est} = \frac{1}{k-1} \sum_{i=1}^{k} \left(s_i^2 - \overline{s^2}\right)^2 \tag{2.3}$$

Relying upon the robustness of Student's t, one could then place confidence intervals around the true σ^2. Clearly the above procedure will be asymptotically valid if k is large. On the other hand, n, and hence k, are usually rather small, so the method may yield rather wide confidence limits.

The <u>jackknife</u> is an alternative grouping procedure. Originally, it was suggested as a means for reducing the bias of a complex non-linear estimate based on a sample; see Quenouille [8], Miller [5], Gaver and Hoel [3], and the recent book by Gray and Schucany [4]. Let e_n be an estimate based on a sample of size n; under many circumstances as n becomes large

$$E[e_n] = a + \frac{b}{n} + \text{remainder}, \tag{2.4}$$

where the remainder is of order $\frac{1}{n^2}$.

An example is, of course, the biased version of the sample variance,

$$e_n = \frac{1}{n} \sum_{i=1}^{n} (x_i - \bar{x})^2. \tag{2.5}$$

Now let $e_{n-1,\, i}$ be the same estimate as e_n, but computed omitting the i^{th} observation, x_i. The <u>pseudo</u> <u>value</u>

$$y_i = n\, e_n - (n-1)\, e_{n-1,\, i} \tag{2.6}$$

has expectation

$$E[y_i] \simeq n\, [a + \frac{b}{n}] - (n-1)\, [a + \frac{b}{n-1}] = a, \tag{2.7}$$

so the leading bias term is removed. One is then led to estimate the number a, the expectation when n becomes large, by the average of the pseudo values, $\bar{y}$. The latter should often be preferable to e_n itself because bias is reduced. This estimator also may have a mean-squared error smaller than that of e_n.

Tukey, see Mosteller and Tukey [6] for details, has suggested that a further step can be taken under benevolent conditions. To achieve these, a preliminary transformation of e_n is often in order: in the case of s^2 one is better off estimating $e_n = \log s^2$. Treat the n pseudo-values as independent, and place confidence limits around a by use of an (approximate) Student's t: refer

$$t = \frac{(\bar{y}-a)\sqrt{n}}{s_y} \tag{2.8}$$

to the t-tables with $n-1$ degrees of freedom.

This idea has been investigated both theoretically and through the use of empirical sampling by R. G. Miller [5]; extensions have been carried out by Arvesen [2]. We shall proceed to show how the latter method compares to the conventional approach when the problems are at least similar to those likely to arise in certain systems studies.

3. Estimation of the Variance of a Skewed Distribution

Imagine that a sample of size $n=10$ is available from a positively skewed distribution. How can we go about estimating the standard deviation, σ^2, and assessing the error of our estimate? As a test case suppose that, unbeknownst to us, the x_i's are sampled from an exponential distribution with unit mean and variance.

We have taken 200 samples of size ten and estimated the standard deviation and confidence limits thereon by use of a (i) conventional Chi-squared procedure, and (ii) the log-transformed jackknife. The estimated actual coverages and average confidence interval widths are table entries, and are to be contrasted to the nominal coverages.

	Nominal Coverage					
	90%	$\hat{E}$[width]	95%	$\hat{E}$[width]	99%	$\hat{E}$[width]
χ^2	70%	0.86	80%	1.1	87%	1.6
Jackknife	80%	3.3	85%	5.2	92%	15.7

The reader is reminded that "coverage" means the probability that a confidence interval actually surrounds the unknown parameter. Nominal coverage refers to the advertised behavior of a particular procedure, which in the present situation is the χ^2. Actual coverages are estimated by simulation.

Clearly the jackknife is more nearly valid than χ^2, although its coverage is low and the typical lengths of the confidence intervals are much wider than those furnished by Chi-squared. Further comparisons are furnished in the subsequent section, which includes discussion of various functionals from a particular stochastic model.

4. Simulation by Antithetic Variables and Variance Estimation

In order to study the time evolution of stochastic models, e.g. for waiting lines and the superficially rather similar rainfall and run-off processes, it is often necessary to resort to computer simulation. For example, consider the following very simple illustrative structure.

$$
\begin{aligned}
W_{t+1} &= W_t + X_t \qquad \text{if} \quad W_t + X_t \geq 0 \\
&= 0 \qquad\qquad\quad \text{if} \quad W_t + X_t < 0
\end{aligned} \tag{4.1}
$$

where X_t is a sequence of random variables. Then $\{W_t\}$ is a random walk with reflecting barrier at zero. Consider these interpretations:

(a) If $X_t = S_t - A_{t+1}$, and S_t is a positive random variable representing the $t^{\underline{th}}$ arrival's service time, while A_{t+1} is the time between arrival t, and arrival t+1, then W_t is the waiting time of the $t^{\underline{th}}$ arrival.

(b) If X_t represents excess (or deficiency) of rainfall in a given year (the $t^{\underline{th}}$), then W_t is accumulated water in a reservoir, say, at time t.

In order to estimate the mean $E[W_t]$ with small variance, it is efficient to generate <u>antithetic pairs</u>, W_t and W_t', and then to average them. That is, we may draw realizations of S_t i=1, 2, 3, ..., and S_t' as follows

$$s_t = F_S^{-1} \; (r\;), \quad \text{and} \quad s_t' = F_S^{-1} \; (1-r\;) \tag{4.2}$$

where r is a uniform pseudo random number. Successive r 's are independent if we believe that $\{S_t\}$ is an independent sequence. The result is that we may generate <u>pairs of</u> realizations

$(U_1, V_1), (U_2, V_2), \ldots (U_n, V_n)$ utilizing antithetic sampling. U_i might for example be W_t, or $\max_{0 \le \tau \le t} W_\tau$, or $\frac{1}{t} \sum_{\tau=1}^{t} W_\tau$, and V_i the same function, computed antithetically. The successive pairs are independent, but U_i and V_i are of course correlated--hopefully negatively. We can now estimate population parameters as follows. Use the statistic W_t as an example.

(a) $E[W_t]$ is estimated by $\bar{W} = \frac{\bar{U} + \bar{V}}{2}$. Confidence limits via Student's t are approximately valid.

(b) Var $[W_t]$ may be estimated by the jackknife.

<u>One</u> procedure is the following.

(b-1) Compute $s^2_{U,-i}$, and $s^2_{V,-i}$ $\quad$ i=1, 2, ... w

These are the sample variances obtained by successively omitting U_1, V_1, then U_2, V_2, <u>etc</u>.

(b-2) Average: $s^2_{-i} = \frac{s^2_{U,-i} + s^2_{V,-i}}{2}$

(b-3) The logarithm, or alternatively the cube root (Wilson-Hilferty) transformation of s^2_{-i} and the corresponding transformation of the average $\frac{s^2_U + s^2_V}{2} = s^2$ then yield up the pseudo values, and finally Student's t generates the confidence limits.

<u>Another</u> procedure is the following.

(b-1') Compute $s^2_{U,-i}$ and $s^2_{V,-i}$.

(b-2') Take logarithms of the above to obtain

$$\overline{\log s^2_{-i}} = \frac{\log s^2_{U,-i} + \log s^2_{V,-i}}{2}.$$

(b-3') Use the results of (b-2') as inputs to the jackknife procedure.

Reasons to prefer one of the above two approaches await further analysis. In this study the former approach was taken.

5. Further Numerical Examples

In (4.1) we let $W_o = 0$, and studied functionals of W_1, W_2, W_3, W_4, W_5. The functionals were

(i) W_5 itself,

(ii) the average of the first five (5.1)

$$\bar{W} = \frac{W_1+W_2+W_3+W_4+W_5}{5}$$

(iii) $\max [W_1, W_2, W_3, W_4, W_5]$.

We considered two cases:

(A) S_n and A_n both exponentially and independently distributed.

(B) S_n constant, A_n exponentially distributed; we then computed constants for realizations of size $n=10$, sometimes without antithetic aid (Straight), and sometimes using the antithetic realization as described (Straight and Antithetic). The following tables illustrate selected results. The purpose was to compare the behavior of χ^2 and the jackknife, both as to coverage compared to advertised nominal coverage and confidence interval width. These tables were computed by Charles Lusky for inclusion in his thesis, thus satisfying part of the requirements for the MS degree at the Naval Postgraduate School. The writer will furnish more detailed sets of tables upon request, but the qualitative conclusions remain essentially the same as those presented below.

Discussion of the Results

(1) Table 1 shows that the conventional χ^2 confidence

limits grossly under-cover the true variance of W_5 when both service and inter-arrival times have exponential distributions. Recall that here, and throughout, sample size is $n=10$.

(2) Table 3 indicates the improvement obtained by use of the jackknife on the same data as that analyzed in Table 1. It is interesting that use of the cube root transformation produces very nearly the same coverage as does the log, but with confidence limits only about one-half as wide on the average and considerably less variable in width as well. This comparison is maintained rather consistently throughout the experiment.

(3) Tables 5 and 6 indicate the further improvement of coverage and shortening of confidence interval widths that occur when antithetic data is available and it is jackknifed as described.

(4) Tables 7 and 8 consider the variance of W_5 when the service times are constant and inter-arrivals are exponential. Coverage conforms well to the nominal. Comparison with Tables 9 and 10 reveals little benefit from combining antithetic realizations in this case.

(5) Tables 11 through 14 compare ordinary with antithetic coverage and confidence interval widths for $\overline{W} = \text{ave}\,(W_1, W_2, \ldots, W_5)$. Coverage is much improved when antithetics are used. Again we notice that the cube root transformation seems to yield narrower limits than does the log, while covering nearly as well.

Conclusions

The results of our sampling experiments indicate that the jackknife is a useful procedure for assessing the variance of a non-

Gaussian distribution. It may be successfully employed to estimate the variance of a response variable in a simulation experiment when the latter experiment utilizes the "antithetic variables" sampling (Monte Carlo) technique. Parenthetically, there is evidence that an appropriate transformation of the function to be estimated enhances the effectiveness of the resulting estimate.

Further investigation of the performance of the jackknife and suggested modifications seems justified in view of the results obtained to date. Almost inevitably, such investigation will involve extensive experimental sampling, plans for which should be guided by whatever theoretical analysis is possible.

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WAITING TIMES   EXP/EXP   ORIG   STRA

STATISTICS ON THE COVERAGE USING THE CHISQUARE PERCENTAGES

RESULTS USING UNTRANSFORMED DATA SUBSETS

PERCENTAGE OF COVERAGE CHI-TABLE PRED.	ACTUAL	DIFFERENCE
80.0	57.0	23.0
90.0	71.5	18.5
95.0	78.5	16.5
98.0	86.5	11.5
99.0	88.5	10.5

Table 1.

AVE WAITING TIMES EXP/EXP ORIG STRA

STATISTICS FOR THE CONFIDENCE INTERVAL WIDTHS AND LIMITS

RESULTS USING UNTRANSFORMED DATA SUBSETS

	EXPECTED VALUES		CONFDNCE INTRVL WIDTH	
CI%	LWR LIMT	UPR LIMT	EXPT VAL	STD DVTN
80	7.14	13.41	6.27	2.56
90	6.66	15.01	8.35	3.41
95	6.28	16.67	10.38	4.24
98	5.88	18.94	13.06	5.34
99	5.64	20.82	15.18	6.20

Table 2.

Table 3.

WAITING TIMES EXP/EXP ORIG STRA

STATISTICS ON THE COVERAGE USING THE JACKKNIFE METHOD

RESULTS USING UNTRANSFORMED JACKKNIFED DATA

PERCENTAGE OF COVERAGE T-TABLE PRED.	ACTUAL	DIFFERENCE
60.0	46.5	13.5
75.0	61.0	14.0
80.0	64.0	16.0
85.0	69.5	15.5
90.0	73.0	17.0
95.0	80.0	15.0
98.0	85.0	13.0
99.0	88.5	10.5

RESULTS USING CUBE ROOT TRANSFORMED JACKKNIFED DATA

PERCENTAGE OF COVERAGE T-TABLE PRED.	ACTUAL	DIFFERENCE
60.0	53.5	6.5
75.0	66.5	8.5
80.0	70.5	9.5
85.0	74.0	11.0
90.0	78.0	12.0
95.0	83.5	11.5
98.0	90.0	8.0
99.0	91.5	7.5

RESULTS USING LOGGED TRANSFORMED JACKKNIFED DATA

PERCENTAGE OF COVERAGE T-TABLE PRED.	ACTUAL	DIFFERENCE
60.0	54.5	5.5
75.0	70.5	4.5
80.0	73.0	7.0
85.0	75.5	9.5
90.0	81.5	8.5
95.0	85.5	9.5
98.0	91.5	6.5
99.0	93.0	6.0

Table 4.

WAITING TIMES EXP/EXP ORIG STRA

STATISTICS FOR THE CONFIDENCE INTERVAL WIDTHS AND LIMITS

RESULTS USING UNTRANSFORMED JACKKNIFED DATA

CI%	EXPECTED VALUES LWR LIMT	EXPECTED VALUES UPR LIMT	CONFDNCE INTRVL WIDTH EXPT VAL	CONFDNCE INTRVL WIDTH STD DVTN
60	13.59	23.86	10.27	7.01
75	11.57	25.88	14.31	9.76
80	10.68	26.77	16.09	10.97
85	9.37	28.07	18.70	12.76
90	8.06	29.38	21.32	14.54
95	5.57	31.88	26.31	17.95
98	2.32	35.13	32.81	22.38
99	-0.18	37.62	37.80	25.79

RESULTS USING CUBE ROOT TRANSFORMED JACKKNIFED DATA

CI%	EXPECTED VALUES LWR LIMT	EXPECTED VALUES UPR LIMT	CONFDNCE INTRVL WIDTH EXPT VAL	CONFDNCE INTRVL WIDTH STD DVTN
60	14.06	27.96	13.90	12.36
75	12.20	31.72	19.52	17.48
80	11.46	33.50	22.04	19.81
85	10.44	36.26	25.82	23.36
90	9.52	39.19	29.67	27.04
95	7.98	45.28	37.29	34.55
98	6.37	54.25	47.88	45.49
99	5.38	61.98	56.60	54.96

RESULTS USING LOGGED TRANSFORMED JACKKNIFED DATA

CI%	EXPECTED VALUES LWR LIMT	EXPECTED VALUES UPR LIMT	CONFDNCE INTRVL WIDTH EXPT VAL	CONFDNCE INTRVL WIDTH STD DVTN
60	14.32	34.28	19.96	24.51
75	12.47	42.52	30.05	39.63
80	11.76	47.03	35.27	48.25
85	10.82	54.88	44.07	64.02
90	9.98	64.57	54.60	84.65
95	8.61	90.10	81.48	144.35
98	7.19	145.61	138.42	293.68
99	6.31	217.93	211.62	514.48

Table 5

WAITING TIMES EXP/EXP ORIG STRA/ANTI

STATISTICS ON THE COVERAGE USING THE JACKKNIFE METHOD

RESULTS USING UNTRANSFORMED JACKKNIFED DATA

PERCENTAGE OF COVERAGE T-TABLE PRED.	ACTUAL	DIFFERENCE
60.0	53.0	7.0
75.0	70.5	4.5
80.0	75.5	4.5
85.0	79.5	5.5
90.0	84.0	6.0
95.0	88.0	7.0
98.0	91.0	7.0
99.0	92.0	7.0

RESULTS USING CUBE ROOT TRANSFORMED JACKKNIFED DATA

PERCENTAGE OF COVERAGE T-TABLE PRED.	ACTUAL	DIFFERENCE
60.0	57.5	2.5
75.0	74.5	0.5
80.0	78.5	1.5
85.0	84.5	0.5
90.0	86.0	4.0
95.0	89.5	5.5
98.0	93.0	5.0
99.0	96.0	3.0

RESULTS USING LOGGED TRANSFORMED JACKKNIFED DATA

PERCENTAGE OF COVERAGE T-TABLE PRED.	ACTUAL	DIFFERENCE
60.0	60.0	0.0
75.0	75.5	-0.5
80.0	78.5	1.5
85.0	85.0	0.0
90.0	88.5	1.5
95.0	91.5	3.5
98.0	94.5	3.5
99.0	96.5	2.5

Table 6.

WAITING TIMES EXP/EXP ORIG STRA/ANTI

STATISTICS FOR THE CONFIDENCE INTERVAL WIDTHS AND LIMITS

RESULTS USING UNTRANSFORMED JACKKNIFED DATA

	EXPECTED VALUES		CONFDNCE INTRVL WIDTH	
CI%	LWR LIMT	UPR LIMT	EXPT VAL	STD DVTN
60	15.57	23.63	8.06	4.28
75	13.99	25.21	11.22	5.96
80	13.29	25.91	12.62	6.71
85	12.26	26.93	14.67	7.80
90	11.24	27.96	16.72	8.89
95	9.28	29.92	20.64	10.97
98	6.73	32.47	25.73	13.68
99	4.77	34.42	29.65	15.76

RESULTS USING CUBE ROOT TRANSFORMED JACKKNIFED DATA

	EXPECTED VALUES		CONFDNCE INTRVL WIDTH	
CI%	LWR LIMT	UPR LIMT	EXPT VAL	STD DVTN
60	15.74	26.08	10.35	7.52
75	14.16	28.62	14.46	10.56
80	13.50	29.80	16.29	11.93
85	12.59	31.59	19.01	13.98
90	11.72	33.47	21.74	16.08
95	10.21	37.27	27.06	20.25
98	8.50	42.69	34.19	26.07
99	7.36	47.23	39.87	30.90

RESULTS USING LOGGED TRANSFORMED JACKKNIFED DATA

	EXPECTED VALUES		CONFDNCE INTRVL WIDTH	
CI%	LWR LIMT	UPR LIMT	EXPT VAL	STD DVTN
60	15.86	29.63	13.77	14.53
75	14.20	34.01	19.81	22.00
80	13.54	36.21	22.67	25.88
85	12.64	39.80	27.17	32.45
90	11.81	43.87	32.06	40.30
95	10.42	53.26	42.84	60.14
98	8.89	69.80	60.90	100.77
99	7.91	87.18	79.26	150.30

Table 7.

WAITING TIMES EXP/CONS ORIG STRA

STATISTICS ON THE COVERAGE USING THE JACKKNIFE METHOD

RESULTS USING UNTRANSFORMED JACKKNIFED DATA

PERCENTAGE OF COVERAGE T-TABLE PRED.	ACTUAL	DIFFERENCE
60.0	66.0	-6.0
75.0	80.0	-5.0
80.0	83.0	-3.0
85.0	86.0	-1.0
90.0	88.5	1.5
95.0	93.0	2.0
98.0	95.0	3.0
99.0	97.0	2.0

RESULTS USING CUBE ROOT TRANSFORMED JACKKNIFED DATA

PERCENTAGE OF COVERAGE T-TABLE PRED.	ACTUAL	DIFFERENCE
60.0	68.5	-8.5
75.0	80.5	-5.5
80.0	87.5	-7.5
85.0	88.5	-3.5
90.0	90.0	0.0
95.0	94.5	0.5
98.0	96.5	1.5
99.0	98.0	1.0

RESULTS USING LOGGED TRANSFORMED JACKKNIFED DATA

PERCENTAGE OF COVERAGE T-TABLE PRED.	ACTUAL	DIFFERENCE
60.0	70.0	-10.0
75.0	82.0	-7.0
80.0	87.0	-7.0
85.0	90.0	-5.0
90.0	90.5	-0.5
95.0	94.0	1.0
98.0	98.5	-0.5
99.0	99.0	0.0

Table 8

WAITING TIMES EXP/CONS ORIG STRA

STATISTICS FOR THE CONFIDENCE INTERVAL WIDTHS AND LIMITS

RESULTS USING UNTRANSFORMED JACKKNIFED DATA

	EXPECTED VALUES		CONFDNCE INTRVL WIDTH	
CI%	LWR LIMT	UPR LIMT	EXPT VAL	STD DVTN
60	8.96	12.77	3.81	1.60
75	8.22	13.52	5.30	2.24
80	7.89	13.85	5.96	2.51
85	7.40	14.33	6.93	2.92
90	6.92	14.82	7.90	3.33
95	5.99	15.74	9.75	4.11
98	4.79	16.95	12.16	5.13
99	3.86	17.87	14.01	5.91

RESULTS USING CUBE ROOT TRANSFORMED JACKKNIFED DATA

	EXPECTED VALUES		CONFDNCE INTRVL WIDTH	
CI%	LWR LIMT	UPR LIMT	EXPT VAL	STD DVTN
60	9.11	13.36	4.26	2.48
75	8.42	14.37	5.94	3.49
80	8.14	14.83	6.69	3.94
85	7.73	15.53	7.80	4.63
90	7.34	16.26	8.92	5.34
95	6.65	17.72	11.07	6.76
98	5.83	19.78	13.95	8.77
99	5.27	21.49	16.22	10.47

RESULTS USING LOGGED TRANSFORMED JACKKNIFED DATA

	EXPECTED VALUES		CONFDNCE INTRVL WIDTH	
CI%	LWR LIMT	UPR LIMT	EXPT VAL	STD DVTN
60	9.20	13.96	4.76	3.89
75	8.53	15.30	6.77	5.91
80	8.26	15.95	7.69	6.96
85	7.88	16.99	9.11	8.75
90	7.52	18.13	10.61	10.90
95	6.88	20.66	13.78	16.32
98	6.16	24.87	18.71	27.39
99	5.66	29.08	23.42	40.72

Table 9

WAITING TIMES EXP/CONS ORIG STRA/ANTI

STATISTICS ON THE COVERAGE USING THE JACKKNIFE METHOD

RESULTS USING UNTRANSFORMED JACKKNIFED DATA

PERCENTAGE OF COVERAGE T-TABLE PRED.	ACTUAL	DIFFERENCE
60.0	68.5	-8.5
75.0	79.5	-4.5
80.0	83.5	-3.5
85.0	89.5	-4.5
90.0	91.5	-1.5
95.0	95.0	0.0
98.0	96.5	1.5
99.0	97.0	2.0

RESULTS USING CUBE ROOT TRANSFORMED JACKKNIFED DATA

PERCENTAGE OF COVERAGE T-TABLE PRED.	ACTUAL	DIFFERENCE
60.0	70.5	-10.5
75.0	83.0	-8.0
80.0	84.5	-4.5
85.0	88.5	-3.5
90.0	93.5	-3.5
95.0	95.5	-0.5
98.0	96.5	1.5
99.0	97.5	1.5

RESULTS USING LOGGED TRANSFORMED JACKKNIFED DATA

PERCENTAGE OF COVERAGE T-TABLE PRED.	ACTUAL	DIFFERENCE
60.0	70.5	-10.5
75.0	83.5	-8.5
80.0	84.5	-4.5
85.0	89.0	-4.0
90.0	93.5	-3.5
95.0	96.5	-1.5
98.0	97.0	1.0
99.0	98.5	0.5

Table 10

WAITING TIMES EXP/CONS ORIG STRA/ANTI

STATISTICS FOR THE CONFIDENCE INTERVAL WIDTHS AND LIMITS

RESULTS USING UNTRANSFORMED JACKKNIFED DATA

CI%	EXPECTED VALUES LWR LIMT	EXPECTED VALUES UPR LIMT	CONFDNCE INTRVL WIDTH EXPT VAL	CONFDNCE INTRVL WIDTH STD DVTN
60	9.45	12.62	3.17	1.08
75	8.83	13.24	4.41	1.50
80	8.55	13.51	4.96	1.69
85	8.15	13.92	5.77	1.96
90	7.74	14.32	6.58	2.24
95	6.97	15.09	8.12	2.76
98	5.97	16.09	10.12	3.44
99	5.20	16.86	11.66	3.97

RESULTS USING CUBE ROOT TRANSFORMED JACKKNIFED DATA

CI%	EXPECTED VALUES LWR LIMT	EXPECTED VALUES UPR LIMT	CONFDNCE INTRVL WIDTH EXPT VAL	CONFDNCE INTRVL WIDTH STD DVTN
60	9.52	13.07	3.55	1.59
75	8.92	13.88	4.96	2.22
80	8.66	14.24	5.58	2.51
85	8.30	14.79	6.49	2.93
90	7.95	15.36	7.41	3.35
95	7.31	16.49	9.18	4.19
98	6.54	18.04	11.51	5.32
99	5.99	19.31	13.32	6.24

RESULTS USING LOGGED TRANSFORMED JACKKNIFED DATA

CI%	EXPECTED VALUES LWR LIMT	EXPECTED VALUES UPR LIMT	CONFDNCE INTRVL WIDTH EXPT VAL	CONFDNCE INTRVL WIDTH STD DVTN
60	9.58	13.51	3.93	2.20
75	8.98	14.50	5.52	3.16
80	8.73	14.96	6.23	3.62
85	8.37	15.68	7.31	4.34
90	8.04	16.44	8.40	5.13
95	7.44	18.03	10.59	6.85
98	6.73	20.40	13.67	9.70
99	6.25	22.50	16.26	12.53

Table 11

AVE WAITING TIMES EXP/EXP ORIG STRA

STATISTICS ON THE COVERAGE USING THE JACKKNIFE METHOD

RESULTS USING UNTRANSFORMED JACKKNIFED DATA

PERCENTAGE OF COVERAGE T-TABLE PRED.	ACTUAL	DIFFERENCE
60.0	42.0	18.0
75.0	53.5	21.5
80.0	59.0	21.0
85.0	64.5	20.5
90.0	68.0	22.0
95.0	74.0	21.0
98.0	79.0	19.0
99.0	82.5	16.5

RESULTS USING CUBE ROOT TRANSFORMED JACKKNIFED DATA

PERCENTAGE OF COVERAGE T-TABLE PRED.	ACTUAL	DIFFERENCE
60.0	47.5	12.5
75.0	58.5	16.5
80.0	62.5	17.5
85.0	71.0	14.0
90.0	74.0	16.0
95.0	82.5	12.5
98.0	86.0	12.0
99.0	88.0	11.0

RESULTS USING LOGGED TRANSFORMED JACKKNIFED DATA

PERCENTAGE OF COVERAGE T-TABLE PRED.	ACTUAL	DIFFERENCE
60.0	49.5	10.5
75.0	62.0	13.0
80.0	66.5	13.5
85.0	73.5	11.5
90.0	78.5	11.5
95.0	84.0	11.0
98.0	88.5	9.5
99.0	91.5	7.5

Table 12.

AVE WAITING TIMES EXP/EXP ORIG STRA

STATISTICS FOR THE CONFIDENCE INTERVAL WIDTHS AND LIMITS

RESULTS USING UNTRANSFORMED JACKKNIFED DATA

	EXPECTED VALUES		CONFDNCE INTRVL WIDTH	
CI%	LWR LIMT	UPR LIMT	EXPT VAL	STD DVTN
60	7.12	12.27	5.15	3.60
75	6.11	13.28	7.18	5.02
80	5.66	13.73	8.07	5.64
85	5.01	14.39	9.38	6.56
90	4.35	15.04	10.69	7.48
95	3.10	16.30	13.20	9.23
98	1.47	17.93	16.46	11.51
99	0.22	19.18	18.96	13.26

RESULTS USING CUBE ROOT TRANSFORMED JACKKNIFED DATA

	EXPECTED VALUES		CONFDNCE INTRVL WIDTH	
CI%	LWR LIMT	UPR LIMT	EXPT VAL	STD DVTN
60	7.36	14.24	6.88	6.65
75	6.42	16.08	9.66	9.43
80	6.05	16.96	10.91	10.71
85	5.54	18.31	12.77	12.65
90	5.07	19.74	14.68	14.69
95	4.28	22.71	18.44	18.87
98	3.44	27.09	23.65	25.06
99	2.91	30.86	27.95	30.50

RESULTS USING LOGGED TRANSFORMED JACKKNIFED DATA

	EXPECTED VALUES		CONFDNCE INTRVL WIDTH	
CI%	LWR LIMT	UPR LIMT	EXPT VAL	STD DVTN
60	7.48	17.70	10.22	17.20
75	6.56	22.24	15.69	30.15
80	6.20	24.82	18.62	38.27
85	5.72	29.48	23.76	54.21
90	5.29	35.50	30.21	76.80
95	4.59	52.66	48.07	150.06
98	3.85	95.32	91.46	363.77
99	3.39	159.07	155.68	723.57

Table 13

AVE WAITING TIMES EXP/EXP ORIG STRA/ANTI

STATISTICS ON THE COVERAGE USING THE JACKKNIFE METHOD

RESULTS USING UNTRANSFORMED JACKKNIFED DATA

PERCENTAGE OF COVERAGE T-TABLE PRED.	ACTUAL	DIFFERENCE
60.0	52.0	8.0
75.0	69.5	5.5
80.0	73.5	6.5
85.0	79.5	5.5
90.0	84.0	6.0
95.0	87.0	8.0
98.0	89.0	9.0
99.0	91.0	8.0

RESULTS USING CUBE ROOT TRANSFORMED JACKKNIFED DATA

PERCENTAGE OF COVERAGE T-TABLE PRED.	ACTUAL	DIFFERENCE
60.0	55.5	4.5
75.0	73.5	1.5
80.0	77.5	2.5
85.0	82.5	2.5
90.0	84.0	6.0
95.0	89.5	5.5
98.0	94.5	3.5
99.0	95.5	3.5

RESULTS USING LOGGED TRANSFORMED JACKKNIFED DATA

PERCENTAGE OF COVERAGE T-TABLE PRED.	ACTUAL	DIFFERENCE
60.0	57.5	2.5
75.0	74.0	1.0
80.0	80.0	0.0
85.0	84.0	1.0
90.0	87.0	3.0
95.0	92.0	3.0
98.0	94.5	3.5
99.0	95.5	3.5

Table 14

AVE WAITING TIMES EXP/EXP ORIG STRA/ANTI

STATISTICS FOR THE CONFIDENCE INTERVAL WIDTHS AND LIMITS

RESULTS USING UNTRANSFORMED JACKKNIFED DATA

CI%	EXPECTED VALUES LWR LIMT	UPR LIMT	CONFDNCE INTRVL WIDTH EXPT VAL	STD DVTN
60	8.11	12.45	4.35	2.23
75	7.25	13.31	6.05	3.10
80	6.88	13.69	6.81	3.49
85	6.32	14.24	7.92	4.05
90	5.77	14.79	9.02	4.62
95	4.71	15.85	11.14	5.70
98	3.34	17.23	13.89	7.11
99	2.28	18.28	16.00	8.19

RESULTS USING CUBE ROOT TRANSFORMED JACKKNIFED DATA

CI%	EXPECTED VALUES LWR LIMT	UPR LIMT	CONFDNCE INTRVL WIDTH EXPT VAL	STD DVTN
60	8.20	13.62	5.42	3.74
75	7.37	14.95	7.58	5.25
80	7.03	15.56	8.53	5.93
85	6.55	16.50	9.95	6.94
90	6.09	17.48	11.38	7.98
95	5.30	19.46	14.16	10.03
98	4.40	22.29	17.89	12.88
99	3.80	24.65	20.85	15.22

RESULTS USING LOGGED TRANSFORMED JACKKNIFED DATA

CI%	EXPECTED VALUES LWR LIMT	UPR LIMT	CONFDNCE INTRVL WIDTH EXPT VAL	STD DVTN
60	8.25	15.23	6.98	7.09
75	7.40	17.40	10.00	10.69
80	7.05	18.48	11.43	12.55
85	6.59	20.24	13.65	15.68
90	6.15	22.21	16.05	19.39
95	5.43	26.70	21.27	28.65
98	4.63	34.42	29.79	47.21
99	4.11	42.34	38.22	69.28

Table 15

MAX WAITING TIMES EXP/EXP ORIG STRA

STATISTICS ON THE COVERAGE USING THE JACKKNIFE METHOD

RESULTS USING UNTRANSFORMED JACKKNIFED DATA

PERCENTAGE OF COVERAGE T-TABLE PRED.	ACTUAL	DIFFERENCE
60.0	46.0	14.0
75.0	58.5	16.5
80.0	61.0	19.0
85.0	64.5	20.5
90.0	71.0	19.0
95.0	81.0	14.0
98.0	86.5	11.5
99.0	89.0	10.0

RESULTS USING CUBE ROOT TRANSFORMED JACKKNIFED DATA

PERCENTAGE OF COVERAGE T-TABLE PRED.	ACTUAL	DIFFERENCE
60.0	46.5	13.5
75.0	60.5	14.5
80.0	65.5	14.5
85.0	72.5	12.5
90.0	78.5	11.5
95.0	87.5	7.5
98.0	91.5	6.5
99.0	95.0	4.0

RESULTS USING LOGGED TRANSFORMED JACKKNIFED DATA

PERCENTAGE OF COVERAGE T-TABLE PRED.	ACTUAL	DIFFERENCE
60.0	47.5	12.5
75.0	64.5	10.5
80.0	70.5	9.5
85.0	76.5	8.5
90.0	82.0	8.0
95.0	89.0	6.0
98.0	94.5	3.5
99.0	95.5	3.5

Table 16

MAX WAITING TIMES EXP/EXP ORIG STRA

STATISTICS FOR THE CONFIDENCE INTERVAL WIDTHS AND LIMITS

RESULTS USING UNTRANSFORMED JACKKNIFED DATA

	EXPECTED VALUES		CONFDNCE INTRVL WIDTH	
CI%	LWR LIMT	UPR LIMT	EXPT VAL	STD DVTN
60	14.22	23.43	9.20	6.15
75	12.42	25.24	12.82	8.56
80	11.62	26.03	14.42	9.63
85	10.45	27.21	16.76	11.19
90	9.27	28.38	19.11	12.76
95	7.04	30.61	23.58	15.74
98	4.12	33.53	29.40	19.63
99	1.89	35.76	33.88	22.62

RESULTS USING CUBE ROOT TRANSFORMED JACKKNIFED DATA

	EXPECTED VALUES		CONFDNCE INTRVL WIDTH	
CI%	LWR LIMT	UPR LIMT	EXPT VAL	STD DVTN
60	14.62	26.11	11.48	9.80
75	12.97	29.05	16.08	13.81
80	12.30	30.44	18.14	15.62
85	11.37	32.56	21.19	18.36
90	10.50	34.79	24.29	21.18
95	9.02	39.38	30.36	26.85
98	7.39	46.03	38.64	34.94
99	6.35	51.68	45.33	41.79

RESULTS USING LOGGED TRANSFORMED JACKKNIFED DATA

	EXPECTED VALUES		CONFDNCE INTRVL WIDTH	
CI%	LWR LIMT	UPR LIMT	EXPT VAL	STD DVTN
60	14.85	29.26	14.41	15.74
75	13.23	34.16	20.93	23.99
80	12.59	36.66	24.08	28.30
85	11.72	40.81	29.10	35.65
90	10.93	45.60	34.67	44.48
95	9.60	56.95	47.35	66.88
98	8.18	77.77	69.59	112.66
99	7.26	100.49	93.23	168.00

Table 17

MAX WAITING TIMES EXP/EXP ORIG STRA/ANTI

STATISTICS ON THE COVERAGE USING THE JACKKNIFE METHOD

RESULTS USING UNTRANSFORMED JACKKNIFED DATA

PERCENTAGE OF COVERAGE T-TABLE PRED.	ACTUAL	DIFFERENCE
60.0	58.0	2.0
75.0	71.0	4.0
80.0	75.5	4.5
85.0	80.5	4.5
90.0	82.5	7.5
95.0	87.5	7.5
98.0	91.5	6.5
99.0	92.5	6.5

RESULTS USING CUBE ROOT TRANSFORMED JACKKNIFED DATA

PERCENTAGE OF COVERAGE T-TABLE PRED.	ACTUAL	DIFFERENCE
60.0	58.0	2.0
75.0	73.5	1.5
80.0	77.0	3.0
85.0	82.0	3.0
90.0	85.0	5.0
95.0	91.5	3.5
98.0	92.5	5.5
99.0	95.0	4.0

RESULTS USING LOGGED TRANSFORMED JACKKNIFED DATA

PERCENTAGE OF COVERAGE T-TABLE PRED.	ACTUAL	DIFFERENCE
60.0	58.0	2.0
75.0	76.0	-1.0
80.0	80.0	0.0
85.0	84.5	0.5
90.0	88.0	2.0
95.0	92.5	2.5
98.0	94.0	4.0
99.0	96.5	2.5

Table 18

MAX WAITING TIMES EXP/EXP ORIG STRA/ANTI

STATISTICS FOR THE CONFIDENCE INTERVAL WIDTHS AND LIMITS

RESULTS USING UNTRANSFORMED JACKKNIFED DATA

	EXPECTED VALUES		CONFDNCE INTRVL WIDTH	
CI%	LWR LIMT	UPR LIMT	EXPT VAL	STD DVTN
60	15.81	23.31	7.50	3.85
75	14.34	24.78	10.44	5.36
80	13.69	25.43	11.74	6.02
85	12.74	26.39	13.65	7.00
90	11.78	27.34	15.56	7.98
95	9.96	29.16	19.21	9.85
98	7.59	31.54	23.95	12.29
99	5.76	33.36	27.59	14.16

RESULTS USING CUBE ROOT TRANSFORMED JACKKNIFED DATA

	EXPECTED VALUES		CONFDNCE INTRVL WIDTH	
CI%	LWR LIMT	UPR LIMT	EXPT VAL	STD DVTN
60	16.00	24.88	8.88	6.01
75	14.60	26.99	12.40	8.43
80	14.01	27.97	13.96	9.51
85	13.18	29.44	16.27	11.13
90	12.39	30.98	18.59	12.78
95	10.99	34.07	23.08	16.03
98	9.37	38.44	29.07	20.51
99	8.26	42.05	33.79	24.18

RESULTS USING LOGGED TRANSFORMED JACKKNIFED DATA

	EXPECTED VALUES		CONFDNCE INTRVL WIDTH	
CI%	LWR LIMT	UPR LIMT	EXPT VAL	STD DVTN
60	16.14	26.71	10.57	9.86
75	14.71	29.72	15.01	14.55
80	14.13	31.19	17.05	16.88
85	13.33	33.51	20.18	20.69
90	12.58	36.05	23.47	25.03
95	11.29	41.61	30.32	35.25
98	9.84	50.61	40.77	53.98
99	8.87	59.24	50.37	74.41

References

[1] Andrews, D. J., Bickel, P. J., Hampel, F. R., Huber, P. J., Rogers, W. H., Tukey, J. W., Robust Estimates of Location, Princeton University Press, 1972.

[2] Arvesen, J., Salsburg, D., "Approximate tests and confidence intervals using the jackknife," Department of Statistics, Purdue University, Mimeo Series No. 267, October 1971.

[3] Gaver, D. P., and Hoel, D., "Comparison of certain small-sample Poisson probability estimates," TECHNOMETRICS, Vol. 12, No. 4, Nov. 1970; pp. 835-850.

[4] Gray, H. L., Schucany, W. R., The Generalized Jackknife Statistic, Marcel Dekker, Inc., 95 Madison Ave., New York, N. Y.

[5] Miller, R. G., "Jackknifing variances," Annals of Mathematical Statistics, Vol. 35, No. 4, 1964, p. 1594.

[6] Mosteller, F., and Tukey, J., Data Analysis, Including Statistics, Chapter 10 in The Handbook of Social Psychology, Vol. II, 2nd Edition, Addison-Wesley, 1968.

[7] Shane, R. M., and Lynn, W. R., "Mathematical model for flood risk evaluation," J. Hydraul, Div., American Society of Civil Engineers, 90 (HYG), p. 1.

[8] Quenouille, M. H., "Notes on bias in estimation," Biometrika, 43, 1956.

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DOCUMENT CONTROL DATA - R & D

Security classification of title, body of abstract and indexing annotation must be entered when the overall report is classified

ORIGINATING ACTIVITY (Corporate author) Naval Postgraduate School Monterey, California 93940	2a. REPORT SECURITY CLASSIFICATION Unclassified 2b. GROUP

REPORT TITLE

Methods for Assessing Variability, with Emphasis on Simulation Data Interpretation

DESCRIPTIVE NOTES (Type of report and inclusive dates)

Technical Report

AUTHOR(S) (First name, middle initial, last name)

Donald P. Gaver

REPORT DATE	7a. TOTAL NO. OF PAGES	7b. NO. OF REFS
November 1972	41	8

CONTRACT OR GRANT NO.	9a. ORIGINATOR'S REPORT NUMBER(S)
PROJECT NO.	9b. OTHER REPORT NO(S) (Any other numbers that may be assigned this report)

DISTRIBUTION STATEMENT

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SUPPLEMENTARY NOTES	12. SPONSORING MILITARY ACTIVITY

ABSTRACT

This report describes and illustrates the use of a grouping technique (the jackknife) for setting confidence limits in simulation situations.

DD FORM 1 NOV 65 1473 (PAGE 1)
S/N 0101-807-6811

14 KEY WORDS	LINK A		LINK B		LINK C	
	ROLE	WT	ROLE	WT	ROLE	WT
Simulation						
Statistics						
Variance						
Waiting lines						
Monte Carlo						